One of my colleagues was stuck with the following question, asked by his IB HL Math student:

Prove that $\sqrt[n]{n!} \leq \frac{n+1}{2}$ for $n \geq 1$.

My colleague tried using Mathematical Induction and was stuck.

After trying this question for a few minutes, I offered the following solution to him:

$$\sqrt[n]{n!} \leq \frac{n+1}{2} \text{ iff } n! \leq \left(\frac{n+1}{2}\right)^n \text{ iff } 2^n n! \leq (n+1)^n$$

Assume $2^n n! \leq (n+1)^n$ is true for some $n = k$, we need to show that $2^{k+1}(k+1)! \leq (k+2)^{k+1}$

LHS of $P_{k+1} = 2^{k+1}(k+1)! \leq 2(k+1)(k+1)^k$ since P_k is true i.e. $2^k k! \leq (k+1)^k$.

Hence we need to show that $2(k+1)(k+1)^k \leq (k+2)^{k+1}$ i.e. $2 \leq \left(\frac{k+2}{k+1}\right)^{k+1}$ i.e. $2 \leq \left(1 + \frac{1}{k+1}\right)^{k+1}$

Let's consider $T_n = \left(1 + \frac{1}{n}\right)^n$ and show that T_n is an increasing sequence.

Clearly $T_1 = 2$ and if T_n is an increasing sequence (DIY) then $T_n \geq 2$. Hence we are done!

However, I'm not satisfied with my proposed solution. My intuition is that there must be a much shorter solution. When my colleague asked this question, it was just before my dinner. After dinner, while I was washing dishes, I thought of a 2 liner solution!

Let's quote the AM-GM inequality:

If $x_i \geq 0 \ \forall \ i = 1,2,3, \ldots, n.$, then $\sqrt[n]{\prod_{i=1}^{n} x_i} \leq \frac{1}{n}\sum_{i=1}^{n} x_i$.

Take $x_i = i \ \forall \ i = 1,2,3, \ldots, n$ and we get $\sqrt[n]{n!} \leq \frac{n+1}{2}$ immediately!

But now the last job to be done is how to prove the AM-GM inequality?

I told my colleague if he is interested, he can find out from my students as I covered it in my HL math class!

Stay tune if you are interested to find out the proof for the AM-GM inequality.

By the way, this is NOT in H2 Math syllabus. For H2 Mathematics, Mathematical Induction only includes series (i.e. summation) and sequence(s). However, this is within IB H2 Math syllabus.

In fact for HL or SL Math exploration, it'll be very nice to 1^{st} prove Hero's formula and then use AM-GM inequality to show that given the perimeter of a triangle, the maximum area formed occurs when it is an equilateral triangle.