Given that $\cos A + \cos B + \cos C = 0$ and $\sin A + \sin B + \sin C = 0$, show that $\cos 2A + \cos 2B + \cos 2C = 0$.

Proof:

Let $a = \cos A + i\sin A, b = \cos B + i\sin B, c = \cos C + i\sin C$.

Note that $a + b + c = (\cos A + \cos B + \cos C) + i(\sin A + \sin B + \sin C) = 0$, and that $|a| = |b| = |c| = 1$.

Now, we give two proofs that $a^2 + b^2 + c^2 = 0$.

<u>Method 1 (my proof):</u>

We treat a, b, c as vectors in an Argand diagram. Since they sum to zero, they form the shape of a triangle in Fig 1.

Since $|a| = |b| = |c|$, this triangle is equilateral, and marked angles x in Fig 1 are all equal to 60°. This means that all angles y are equal to 120°

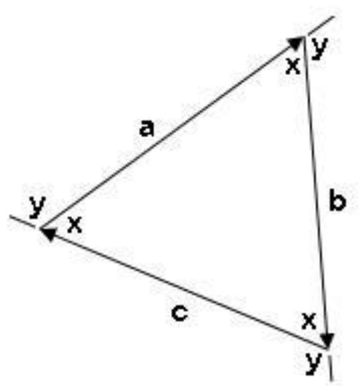

Figure 1

Now we treat a, b, c as points in the Argand diagram. Note that angles y marked in Fig 2 correspond to angles y in Fig 1, thus a, b, c form an equilateral triangle, having the origin as centre.

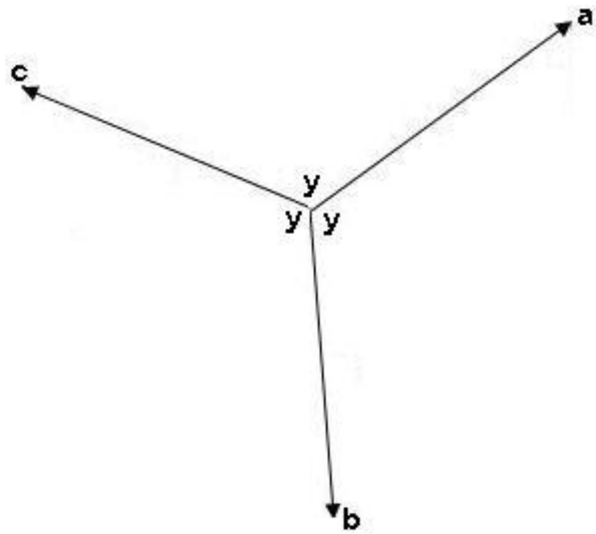

Figure 2

Let $\omega = e^{\frac{2\pi i}{3}}$ be a primitive cube root of unity. Recall that $1, \omega, \omega^2$ forms an equilateral triangle, and multiplication by a is a rotation since $|a| = 1$.

a, b, c form an equilateral triangle. Thus,

$$\{a, b, c\} = \{a, a\omega, a\omega^2\}.$$

Keeping in mind that $\omega^3 = 1$, we have

$$\{a^2, b^2, c^2\} = \{a, a\omega, a\omega^2\}$$

(Graphically, this means that a^2, b^2, c^2 form an equilateral triangle)

$\omega^3 - 1 = 0$ is equivalent to $(\omega - 1)(\omega^2 + \omega + 1) = 0$. Since $\omega \neq 1$, $\omega^2 + \omega + 1 = 0$.

As $\{a^2, b^2, c^2\} = \{a, a\omega, a\omega^2\}$, we have

$$a^2 + b^2 + c^2 = a + a\omega + a\omega^2 = a(1 + \omega + \omega^2) = 0.$$

<u>Method 2 (from an Olympiad book):</u>

$a + b + c = 0$ implies that $a^* + b^* + c^* = 0$. (conjugates)

Since $|a| = 1$, we have $aa^* = |a|^2 = 1$ and hence $a^* = \frac{1}{a}$.

Similarly, $b^* = \frac{1}{b}, c^* = \frac{1}{c}$.

Thus, $a^* + b^* + c^* = 0$ is equivalent to $\frac{1}{a} + \frac{1}{b} + \frac{1}{c} = 0$.

$$0=\frac{1}{a}+\frac{1}{b}+\frac{1}{c}=\frac{ab+bc+ca}{abc}\Rightarrow ab+bc+ca=0.$$
$$0=(a+b+c)^2-2(ab+bc+ca)=a^2+b^2+c^2.$$

Thus, each method gives $a^2+b^2+c^2=0$.

Since the arguments of a,b,c are A, B, C, the arguments of a^2,b^2,c^2 are 2A, 2B, 2C.

Taking real parts of $a^2+b^2+c^2=0$ gives

$\underline{\cos 2A+\cos 2B+\cos 2C=0}$. QED.

(Taking imaginary parts of $a^2+b^2+c^2=0$ gives $\sin 2A+\sin 2B+\sin 2C=0$)

Note: A generalization would be

$\cos nA+\cos nB+\cos nC=\sin nA+\sin nB+\sin nC=0$, for all n not divisible by 3.

Proof:

From Proof 1, $\{a,b,c\}=\{a,a\omega,a\omega^2\}$. Thus,

$$\{a^n,b^n,c^n\}=\{a,a\omega^n,a\omega^{2n}\}.$$

Since n is not divisible by 3, it leaves a remainder of 1 or 2.
If n leaves remainder 1 upon division by 3, then 2n leaves remainder 2.
If n leaves remainder 2 upon division by 3, then 2n leaves remainder 1.

In either case, since $\omega^3=1$

$$\{a^n,b^n,c^n\}=\{a,a\omega^n,a\omega^{2n}\}=\{a,a\omega,a\omega^2\}$$

(Graphically, this means that a^n,b^n,c^n form an equilateral triangle)

Thus, $a^n+b^n+c^n=a+a\omega+a\omega^2=a(1+\omega+\omega^2)=0$. Taking real and imaginary parts gives the result.

Contributed by Ang Jie Jun