Xu Na

She joined ACJC in Jan 1996 and excelled in Math C and F Math. Her L1R5 was 15 (She had C6 in O Level English, and her humanities subject should be C5).

She was posted out of ACJC after the posting results were released and she approached me to help her appeal to stay on in ACJC.

I brought her to the General Office at once, where all the appealing students (and parents) were waiting to see the Principal. It so happened that the student counselor on duty was my student (Adrian, from SB1) and he told me to go in right after a student came out of the Principal Office.

I went in together with Xu Na and the Principal (Mr Wan) commented, “Mr Goh, you cut the queue”. I didn’t comment on that because I just wanted to focus on helping her. Then Mr Wan said,”Another case of being weak in English.” I could feel that Xu Na was embarrassed, but she felt even more apologetic towards me. Mr Wan approved the appeal on the spot with one condition: Xu Na would only be offered to take 3 A Level subjects. Xu Na chose Math C, Further Math and Physics.

She did well in the JC1 promotional examination and was allowed to take Math C and Physics S-papers. Subsequently she approached me, as she needed to go to the British Council for English tuition, but it clashed with the Math C S-paper session. As I was conducting that particular Math C S-paper lesson, I approved and told her to see me another time if she needed extra help with that topic.

At A Levels, she scored distinctions in all her 3 A Level subjects and Math C S-paper. She also obtained a merit for her Physics S-paper.

Few years later, in 1999, I was at the NUS Science Canteen for food after my blood donation at NUH. I bumped into Xu Na there, and asked her why she didn’t score distinction for her Physics S-paper as her Physics teacher expected her to do very well for it. She told me that her scientific calculator wasn’t working that day. I was speechless as I was the presiding examiner for her batch, and her Physics teacher was the other presiding examiner. We would have definitely help her to get a replacement if we knew about it. So for her Physics S paper she completed with manual calculation.

She was an introvert and preferred not to trouble others as far as possible.
Every time when she approached me to clarify math concepts or to solve tough math questions, she was shy and humble and always worried about wasting my time.

In one of the lectures on inequality, I showed the 2 particular cases of proofs for Arithmetic - Geometric inequality when $n = 2$ and extended it to $n = 4$. I told the students I haven't found a way to prove the case where $n = 3$. About a week later, Xu Na shared with me how she proved the case for $n = 3$.

Arithmetic- Geometric inequality (AM-GM $\neq$)

For all $x_k \geq 0, 1 \leq k \leq n$ & $k \in \mathbb{N}$, $\quad \frac{1}{n}\sum_{k=1}^{n} x_k \geq \sqrt[n]{\prod_{k=1}^{n} x_k}$.

In particular, $\frac{1}{n}\sum_{k=1}^{n} x_k = \sqrt[n]{\prod_{k=1}^{n} x_k} \iff x_1 = x_2 = \ldots = x_n$.

When $n = 2$:

$$\left(\sqrt{x_1} - \sqrt{x_2}\right)^2 \geq 0 \iff \frac{x_1 + x_2}{2} \geq \sqrt{x_1 x_2}$$

Clearly, $\frac{x_1+x_2}{2} = \sqrt{x_1 x_2} \iff x_1 = x_2$

When $n = 4$:
Let $y_1 = \frac{x_1+x_2}{2}$ & $y_2 = \frac{x_3+x_4}{2}$.

In the following LHS handles the inequality and RHS handles the equality :

$$\left(\sqrt{y_1} - \sqrt{y_2}\right)^2 \geq 0 \iff \frac{y_1 + y_2}{2} \geq \sqrt{y_1 y_2}$$
$$\frac{\left(\frac{x_1 + x_2}{2}\right) + \left(\frac{x_3 + x_4}{2}\right)}{2} \geq \sqrt{\left(\frac{x_1 + x_2}{2}\right)\left(\frac{x_3 + x_4}{2}\right)}$$
$$\frac{x_1 + x_2 + x_3 + x_4}{4} \geq \sqrt{\sqrt{x_1 x_2}\sqrt{x_3 x_4}}$$
$$\frac{x_1 + x_2 + x_3 + x_4}{4} \geq \sqrt[4]{x_1 x_2 x_3 x_4}$$

$$\frac{x_1 + x_2}{2} = \sqrt{x_1 x_2} \iff x_1 = x_2 \ \&$$
$$\frac{x_3 + x_4}{2} = \sqrt{x_3 x_4} \iff x_3 = x_4$$
$$\frac{y_1 + y_2}{2} = \sqrt{y_1 y_2} \iff y_1 = y_2$$
$$\iff \left(\frac{x_1 + x_2}{2}\right) = \left(\frac{x_3 + x_4}{2}\right)$$
$$\frac{x_1 + x_2 + x_3 + x_4}{4} = \sqrt{\sqrt{x_1 x_2}\sqrt{x_3 x_4}}$$
$$= \sqrt[4]{x_1 x_2 x_3 x_4}$$
$$\iff x_1 = x_2 = x_3 = x_4$$

Xu Na shared: Replace x_4 by $\frac{1}{3}(x_1 + x_2 + x_3)$

So I came out with the following:

$$\frac{x_1 + x_2 + x_3 + \frac{1}{3}(x_1 + x_2 + x_3)}{4} \geq \sqrt[4]{(x_1 x_2 x_3)\frac{1}{3}(x_1 + x_2 + x_3)}$$
$$\left(\frac{x_1 + x_2 + x_3}{3}\right)^{3/4} \geq \sqrt[4]{x_1 x_2 x_3}$$
$$\frac{x_1 + x_2 + x_3}{3} \geq \sqrt[3]{x_1 x_2 x_3}$$

$$x_4 = \frac{1}{3}(x_1 + x_2 + x_3)$$
$$x_1 = x_2 = x_3 = \frac{1}{3}(x_1 + x_2 + x_3)$$
$$x_1 = x_2 = x_3$$

Let's proceed to prove the general AM-GM $\neq$, but let's be convinced that it is worth the trouble.

Example 1

If $y = x^2(1-x)$, find the local maximum point if $0 < x < 1$.

By using AM-GM $\neq$,

$$\frac{\frac{x}{2}+\frac{x}{2}+(1-x)}{3} \geq \sqrt[3]{\left(\frac{x}{2}\right)^2(1-x)}$$

$$\left(\frac{x}{2}\right)^2(1-x) \leq \left(\frac{1}{3}\right)^3$$

$$x^2(1-x) \leq \frac{4}{27}$$

Hence when $\frac{x}{2} = \frac{x}{2} = (1-x)$, max $y = \frac{4}{27}$. Thus the local max point is at $\left(\frac{2}{3}, \frac{4}{27}\right)$.

The trick is to make the numerator of the LHS to be a constant. This method doesn't need to use calculus.

Example 2

Given the perimeter of a triangle, show that the maximum area occurs when it is an equilateral triangle.

We shall assume that if a, b & c are the lengths of a triangle & if $s = \frac{a+b+c}{2}$ then area of the triangle is

$$\sqrt{s(s-a)(s-b)(s-c)}.$$

This formula can be proven at least 3 ways*.

Note that s is a constant.

$$\frac{(s-a)+(s-b)+(s-c)}{3} \geq \sqrt[3]{(s-a)(s-b)(s-c)}$$

$$\sqrt{(s-a)(s-b)(s-c)} \leq \left(\sqrt{\frac{s}{3}}\right)^3$$

$$\sqrt{s(s-a)(s-b)(s-c)} \leq \frac{s^2}{3\sqrt{3}}$$

Maximum area occurs when $(s-a) = (s-b) = (s-c)$ ie. $a = b = c$.

Maximum area $= \frac{s^2}{3\sqrt{3}}$ when $a = b = c$ i.e it is an equilateral triangle.

Let P be the perimeter of the triangle ie $P = 2s$, hence maximum area $= \frac{P^2}{12\sqrt{3}}$ sq units.

If we solve this question using calculus, it is tedious and we need to apply partial differentiation. Alternatively it can also be done using quadratic idea however the expressions are long and working is tedious.

*There are 2 ways to prove this formula using trigonometry and another by using complex numbers:

1. Refer to ‘A’ level Math exam N94/I/20.
2. Refer to my student’s (Tri) proof: https://www.singaporeibmathstuition.com/free-resources/heron-formula-by-tri/
3. Using complex numbers please refer to the following: https://mathgarage.wordpress.com/wp-content/uploads/2013/01/heron_by_complex_number.gif

Example 3

Prove that $\sqrt[n]{n!} \le \frac{n+1}{2}\ \forall\ n \ge 1$.

Let’s recall AM-GM $\neq$.

For all $x_k \ge 0, 1 \le k \le n\ \&\ k \in \mathbb{N}$,
$$\frac{1}{n}\sum_{k=1}^{n} x_k \ge \sqrt[n]{\prod_{k=1}^{n} x_k}\ .$$

In particular,
$$\frac{1}{n}\sum_{k=1}^{n} x_k = \sqrt[n]{\prod_{k=1}^{n} x_k} \Leftrightarrow x_1 = x_2 = \ldots = x_n.$$

Let’s take $x_m = m$ where $m = 1,2,3,\ldots,n$ then by applying AM-GM $\neq$:

$$\sqrt[n]{\prod_{m=1}^{n} m} \le \frac{1}{n}\sum_{m=1}^{n} m$$

$$\sqrt[n]{n!} \le \frac{1}{n}\left[\frac{n}{2}(n+1)\right] = \frac{n+1}{2}\ \text{(shown)}$$

For alternative solution please refer to: https://www.singaporeibmathstuition.com/articles/appln-of-am-gm-for-an-mi-qn-ib-hl-math/

In order to prove by induction that the AM-GM $\neq$ is true:

Steps:

1. $P(1)$ is true (base case)
2. For all $m \in \mathbb{Z}^+$, show that $P(m) \Rightarrow P(2m)$ & $P(m) \Rightarrow P(m-1)$ (inductive steps).

Clearly $P(1)$ is true and we’ve shown $P(2)$ is also true and by applying $P(m) \Rightarrow P(2m)$ ie $P(4)$ is true, then since $P(m) \Rightarrow P(m-1)$ means $P(3)$ is true. Basically using $P(m) \Rightarrow P(2m)$ we obtain $P(2^n)$ are true. Then the cases between $P(2^{n-1})$ & $P(2^n)$ we use $P(m) \Rightarrow P(m-1)$ to show that they are all true too.

How to show that $P(m) \;\Rightarrow\; P(2m)$?
Assume that $P(m)$ is true for some m, we would like to show that $P(2m)$ is also true:

$$\frac{1}{m}\sum_{k=1}^{m} x_k \geq \sqrt[m]{\prod_{k=1}^{m} x_k}\,.$$

Using $x_k = \frac{1}{2}(y_{2k-1} + y_{2k})$:
To prove the inequality:

$$\text{LHS of } P(2m) \quad = \frac{1}{m}\sum_{k=1}^{m}\frac{1}{2}(y_{2k-1} + y_{2k}) \;\geq\; \sqrt[m]{\prod_{k=1}^{m}\left[\frac{1}{2}(y_{2k-1} + y_{2k})\right]}$$

$$\frac{1}{2m}\sum_{k=1}^{2m} y_k \;\geq\; \sqrt[m]{\prod_{k=1}^{m}\left[\sqrt{y_{2k-1}y_{2k}}\right]}$$

$$= \sqrt[2m]{\prod_{k=1}^{2m} y_k} = \text{RHS of } P(2m)$$

To prove the equality:

$$\frac{1}{2}(y_{2k-1} + y_{2k}) = \sqrt{y_{2k-1}y_{2k}} \Leftrightarrow y_{2k-1} = y_{2k}$$

Observe that $\displaystyle \frac{1}{m}\sum_{k=1}^{m}\frac{1}{2}(y_{2k-1} + y_{2k}) = \frac{1}{2m}\left[(y_1 + y_{2m}) + \sum_{k=1}^{m-1}(y_{2k} + y_{2k+1})\right]$

$$\frac{1}{2m}\sum_{k=1}^{2m} y_k \;=\; \sqrt[m]{\sqrt{y_1 + y_{2m}}\prod_{k=1}^{m-1}\left[\sqrt{y_{2k}y_{2k+1}}\right]}$$

$$= \sqrt[m]{\prod_{k=1}^{m}\left[\sqrt{y_{2k-1}y_{2k}}\right]}$$

$$= \sqrt[2m]{\prod_{k=1}^{2m} y_k}$$

$$\tfrac{1}{2}(y_1 + y_{2m}) = \sqrt{y_1 y_{2m}} \Leftrightarrow y_1 = y_{2m} \;\&$$

$$\frac{1}{2}(y_{2k} + y_{2k+1}) = \sqrt{y_{2k}y_{2k+1}} \Leftrightarrow y_{2k} = y_{2k+1} \text{ for } 1 \leq k \leq m-1$$

Since $\frac{1}{2}(y_{2k-1} + y_{2k}) = \sqrt{y_{2k-1}y_{2k}} \Leftrightarrow y_{2k-1} = y_{2k}$

Hence, $y_1 = y_2 = \ldots = y_{2m}$.

How to show that $P(m) \;\Rightarrow\; P(m-1)$? I extended Xu Na's idea for the case of proving $n = 4$ to $n = 3$.

Assume that $P(m)$ is true for some m, we would like to show that $P(m-1)$ is also true:

For $m > 1$, $\quad x_m = \frac{1}{m-1}\sum_{k=1}^{m-1} x_k$ & $x_1 = x_2 = \ldots = x_{m-1}$

To prove the inequality:

$$\frac{1}{m}\sum_{k=1}^{m} x_k \geq \sqrt[m]{\prod_{k=1}^{m} x_k}$$

$$\left(\frac{1}{m}\sum_{k=1}^{m-1} x_k\right) + \frac{1}{m(m-1)}\sum_{k=1}^{m-1} x_k \geq \sqrt[m]{\left(\prod_{k=1}^{m-1} x_k\right)\left(\frac{1}{m-1}\sum_{k=1}^{m-1} x_k\right)}$$

$$\frac{1}{m-1}\sum_{k=1}^{m-1} x_k \geq \sqrt[m]{\left(\prod_{k=1}^{m-1} x_k\right)\left(\frac{1}{m-1}\sum_{k=1}^{m-1} x_k\right)}$$

$$\left(\frac{1}{m-1}\sum_{k=1}^{m-1} x_k\right)^{1-\frac{1}{m}} \geq \sqrt[m]{\left(\prod_{k=1}^{m-1} x_k\right)}$$

$$\frac{1}{m-1}\sum_{k=1}^{m-1} x_k \geq \sqrt[m-1]{\prod_{k=1}^{m-1} x_k}$$

To prove the equality:

From $P(m)$: $\quad x_1 = x_2 = \ldots = x_m$

Note that $\quad x_m = \frac{1}{m-1}\sum_{k=1}^{m-1} x_k = \frac{1}{m-1}[(m-1)x_1] = x_1$

Hence, $\quad x_m = \frac{1}{m-1}\sum_{k=1}^{m-1} x_k \quad$ is a consistent with $P(m)$ where $x_1 = x_2 = \ldots = x_m$

Hence, we still have $x_1 = x_2 = \ldots = x_{m-1}$.

Hence, $P(m) \Rightarrow P(m-1)$

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Thus we've shown the 2 inductive steps $P(m) \Rightarrow P(2m)$ & $P(m) \Rightarrow P(m-1)$ are true for $m > 1$.

I'm Mr Goh Tiong Gee. I've the privilege to be part of Xu Na's learning journey. It was a pain to lose her in such a manner. I've seen some unkind remarks about her and I feel upset. That is why I approached media to share how lucky to have such a sensible student.

Reports about Xu Na (English version)
https://www.facebook.com/share/p/19yQfUGWbS/

有关徐娜的华文报道
https://www.zaobao.com.sg/news/singapore/story20251010-7645549